

## Pin-Hole Water Flow from Cylindrical Bottles

Paulo Murilo Castro de Oliveira, Antonio Delfino,  
Eden Vieira Costa and Carlos Alberto Faria Leite

Instituto de Física, Universidade Federal Fluminense  
av. Litorânea s/n, Boa Viagem, Niterói RJ, Brazil 24210-340  
e-mail PMCO @ IF.UFF.BR

**Abstract** We performed an experiment on elementary hydrodynamics. The basic system is a cylindrical bottle from which water flows through a pin-hole located at the bottom of its lateral surface. We measured the speed of the water leaving the pin-hole, as a function of both the time and the current level of water still inside the bottle. The experimental results are compared with the theory. The theoretical treatment is a very simple one based on mass and energy conservation, corresponding to a widespread exercise usually adopted in university basic disciplines of Physics.

We extended the previous experiment to another similar system using two equal bottles with equal pin-holes. The water flowing from the first bottle feeds the second one located below it. The same concepts of mass and energy conservation now lead to a non-trivial differential equation for the lowest bottle dynamics. We solved this equation both numerically and analytically, comparing the results with the experimental data.

## I — Introduction

Many university textbooks (see, for instance, [1–7]) refer to the problem of water squirted freely from a small hole in a bottle, as in figure 1. The water speed  $V$  at the hole depends on the height  $H$  of the free liquid surface above the pin-hole, according to the so-called Torricelli’s law

$$V^2 = 2gH \quad , \quad (1)$$

where  $g$  represents the earth gravitational constant. This result was obtained by Galileo’s assistant E. Torricelli, in 1636 [7]. It is normally obtained nowadays through Bernoulli’s theorem

$$p + \frac{1}{2}\rho V^2 + \rho gy = C \quad , \quad (2)$$

introduced by Daniel Bernoulli in his book on hydrodynamics of 1738, a century later than Torricelli’s law. Here,  $\rho$  represents the liquid density, while  $p$ ,  $V$  and  $y$  are respectively the local hydrostatic pressure, speed and height, all of them measured at the same point inside the liquid. The constant  $C$  means that the sum on the left-hand side is the same, independent of the particular point where the measurements are performed. In our case, one must take two points: the first at the free liquid surface inside the bottle, with  $y = H$ , neglecting the speed there; and the second point just after the pin-hole, with  $y = 0$ , outside the bottle. The hydrostatic pressure at these two points is the same, namely the atmospheric one. Equation (1) follows directly. This theorem holds for stationary liquid flows, and comes from energy conservation arguments. Thus, in adopting it one is neglecting any energy loss due to viscosity, turbulence, etc.

Because the downward speed of the free liquid surface inside the bottle was neglected, there is a missing factor of  $(1 - a^2/A^2)$  on the left-hand side of equation (1), where  $A$  and  $a$  represent the cross sections of the bottle and the liquid vein, respectively. This correction comes from equation (7) to be introduced later on. The cylindrical bottle used in our experiment has a diameter of 9.5cm, and the pin-hole is made of a thin plastic tube (extracted from a ball pen) whose diameter is 2.0mm. Thus, the neglected factor of  $(1 - a^2/A^2)$  represents a relative deviation of the order of  $10^{-7}$ , much smaller than our experimental accuracy. Also, the liquid vein cross section  $a$  is a little bit smaller than that of the pin-hole, due to the phenomenon of *vena contracta* [3,5,7].

The text is organized as follows. In section II we describe our first, simplest experiment with a single bottle (figure 1), where we measured simultaneously the speed  $V$  and the time  $T$  spent since the initial height  $H_0$ , while the bottle drains off, for successive values of the

decreasing height  $H$ . The speeds are indirectly obtained through the range  $X$  measured on the horizontal rule located at a fixed height below the bottle:  $X$  is thus proportional to  $V$ . In order to compare our experimental results with the theory, a simple correction is made on Torricelli's law, taking energy losses into account. We also obtain the solution of Bernoulli's differential equation (2) for this case, giving the time evolution of  $V(T)$  and  $H(T)$  during the flow, comparing them with the experiment. In section III we describe our second experiment with two identical bottles (figure 2), the water squirted from the first one being caught by the other through a funnel. Initially, the bottles are equally filled, and then both flows start at the same time. Since we have already verified the validity of Torricelli's relation with good accuracy from the first experiment, we used it in the second one, instead of measuring the bottom velocity  $V'$ . Our accuracy in measuring  $H$  (or  $H'$ ) is better than the corresponding one for  $X$  (or  $X'$ , or  $V'$ ). Thus, in this second experiment we measured only the time evolution of  $H'(T)$ . In this case, the analytical solution for Bernoulli's differential equation is not trivial, and we solved it numerically. Nevertheless, the analytical solution could be obtained by using some tricks. The results are compared with the experiment. In section IV we present our conclusions.

## II — One-Bottle Experiment

The water is squirted through a horizontal thin plastic tube installed at the pin-hole. A (parallel) rule is placed below it at a distance  $Y_0$ . Thus, the falling time of a small piece of water is  $\sqrt{2Y_0/g}$ . This is the constant of proportionality between the measured range  $X$  in figure 1 and the speed  $V$ , i.e.  $V = X/\sqrt{2Y_0/g}$ . The water flow is started by one of the authors, responsible for measuring heights, from an initial value  $H_0 = 20.0\text{cm}$ , at time  $T = 0$ . At this same instant, another author starts a chronometer with memories (actually, a computer program which stores the time spent since  $T = 0$  each time some key is stroked), and a third author reads  $X$  at the rule. A fourth author books this value. The whole procedure is repeated when the liquid level reaches the mark of  $H = 19.5\text{cm}$ , again at  $H = 19.0\text{cm}$ , then at  $H = 18.5\text{cm}$ , and so on. At the end, one has a three-column table with  $H$ ,  $T$  and  $X$  (this last being proportional to the speed  $V$ ). The initial range  $X_0 = (28.0 \pm 0.3)\text{cm}$  was measured for  $Y_0 \cong 11\text{cm}$  (for the sake of clarity, figure 1 shows a larger value, but our actual measures were all taken with  $Y_0 \cong 11\text{cm}$ ), corresponding to an initial speed  $V_0 \cong 1.9\text{m/s}$ , supposing  $g \cong 9.8\text{m/s}^2$ . However, one does not need to perform the transformation from  $X$  to  $V$ : we will use always  $X$  instead of  $V$ . In doing so, we do not care about precise measures of  $Y_0$  and  $g$ .

Figure 3 shows the plot of the squared range  $X^2$  *versus* the height  $H$ , which must be

a straight line, namely  $X^2 = 4Y_0H$ , according to Torricelli's law, equation (1). However, our experimental straight line does not cross the origin! On the contrary, the best linear fit for our data, the continuous line  $X^2 = K(H - R)$  with optimum values for  $K$  and  $R$ , crosses the  $H$  axis at a minimum residual height  $R = 2.05\text{cm}$ . Indeed, during the experiment, we noted that the continuous water flow ceases *before* the surface level inside the bottle reaches the pin-hole. This precociously interrupted flow occurs when the height is still around 2cm. At this situation, the surface tension inside the thin tube is enough to compensate alone the overpressure due to the residual level height, except for isolated drops which starts to appear after this point. Even before this, for heights below 5cm, we could note that the water flow leaving the pin-hole is no longer completely stable as it was for the initial heights above 5cm: some oscillations and instabilities appear below that height, as commented later on.

Anyway, equation (1) is not supposed to fit the experimental data, for which we cannot neglect energy losses. Indeed, the sum  $C$  in equation (2) is not strictly the same for all positions inside the liquid, but must be smaller at the pin-hole exit than inside the bottle. This behavior is expected due to the viscosity and turbulences leading to energy losses when the liquid passes through the thin tube. When applying Bernoulli's equation (2) for a point at the free liquid surface and another point at the pin-hole exit, one must discount some energy density amount from the former one. We will assume here that this amount is the same during all the flow, which corresponds to subtract a constant from the kinetic energy term in equation (2), or, alternatively, subtract a constant height  $R$  from the actual height  $H$ . With this correction, Torricelli's law becomes

$$V^2 = 2g(H - R) \quad , \quad (3)$$

instead of equation (1), where the residual height  $R$  is a constant. Indeed, our experimental data agree quite well with this modified form of Torricelli's law, equation (3), as can be appreciated in figure 3. Also from figure 3, one can measure the slope  $K = 4Y_0 = 43.36\text{cm}$ , in complete agreement with the rule's vertical distance  $Y_0 = 10.8\text{cm}$ .

The subtracted term  $R = 2.05\text{cm}$  corresponds to an energy density  $\rho g R$  dissipated at the thin tube, i.e. the lost energy per unit volume of liquid. Thus, considering the outflow rate  $aV$ , i.e. the volume of liquid crossing the tube per unit time, one obtains an energy dissipation rate, i.e. a dissipated power  $\rho g a R V$  proportional to the speed  $V$ . This energy dissipation comes from the fact that different points correspond to different liquid speeds: small pieces of water near the internal walls, inside the thin tube, move slower than those near the tube axis. The relative displacement between adjacent liquid layers dissipates the energy (cylindrical layers inside the thin tube). The constancy of  $R$  during the whole flow, experimentally supported by the straight line behavior in figure 3, means that the energy

dissipation occurs according to a constant resultant viscous force  $\rho g a R \approx 6 \times 10^{-4} \text{N}$ . Of course, the particular value of this force must depend on the liquid's viscosity, and also on the characteristics of the thin tube itself, e.g. its diameter and length, the material of which it is made, etc. Thus, the value of this constant must change for another experimental apparatus. In order to test this, we performed the same experiment once again, by using another bottle, now with an inner diameter of 5.1cm and initial height  $H_0 = 42 \text{cm}$ , also with another pin-hole. The experimental results are shown in figure 4. The straight line behavior agrees again with our modified Torricelli's law, equation (3). The thin tube is now metallic, with an inner diameter of 3.5mm. Accordingly, the new value  $R = 4.81 \text{cm}$  corresponds to a larger resulting viscous force of  $5 \times 10^{-3} \text{N}$ , approximately 7 times larger than that of our first bottle. However, this new value is again constant during the whole flow. For this new bottle, we start to observe oscillations and instabilities of the outflow below  $H \approx 12 \text{cm}$ . For  $H \approx 8 \text{cm}$  these instabilities become strong enough to forbid accurate measurements of  $X$ .

It is possible to make our results independent of the particular geometry of the (cylindrical) bottle used during the experiment, provided the condition  $a \ll A$  remains valid. In order to do such an universal (bottle-independent) analysis, instead of the quantities  $H$  and  $V$  denoted by capital letters, we will adopt the following lowercase, dimensionless, reduced variables

$$h = \frac{H - R}{H_0 - R} \quad , \quad (4)$$

and

$$v = \frac{V}{V_0} = \frac{X}{X_0} \quad . \quad (5)$$

With these definitions, the modified Torricelli's law, equation(3), reads simply

$$v = \sqrt{h} \quad . \quad (6)$$

Figure 5 shows a plot of the experimental values of  $v$  *versus*  $h$ , as well as the continuous line corresponding to the theoretical equation (6). Data from both bottles, presented first separately in figures 3 and 4, are now superimposed into the same plot, defining a single, universal, bottle-independent curve. Except for small deviations at the end of the flow, near the origin in figure 5, the agreement between theory and experiment is quite good. A better test for Torricelli's law is shown in figure 6, where the same experimental data were plotted in a  $\log \times \log$  scale. The continuous line is the best linear fit for our first bottle (circles), excluding data taken for  $H < 5 \text{cm}$ , where the quoted flow instabilities and

oscillations start to be noted by us during the experiment. The slope of 0.497 we obtained must be compared with the exponent  $1/2$  of the square root appearing in equation (6). For the other bottle (crosses), excluding  $H < 12\text{cm}$ , we obtained a slope of 0.501.

Hereafter, all our remainder experimental results correspond to the bottle geometry shown in figure 1, because the measures are more accurate in this case (note the number of experimental points in figure 3, more than twice the corresponding number in figure 4). When the liquid vein crosses the rule height  $Y_0$  below the pin-hole, it is not completely stable, presenting some fluctuations as visible in figure 1. In measuring  $X$ , we are forced to estimate an “average” value at each measuring time. Our actual measurements were made at a vertical height  $Y_0 \cong 11\text{cm}$ , smaller than that shown in figure 1, in order to minimize these fluctuations. Indeed, the flow seems very stable near the pin-hole, at least for large values of the height  $H$ . Once we have already a good agreement between theory and experiment, concerning the modified Torricelli’s law, hereafter we will use equations (4) and (6) in order to obtain the reduced velocity  $v$  from the actually measured height  $H$ , instead of equation (5) with the measured horizontal range  $X$ . The reason for that is the better accuracy we have in measuring  $H$  at the marks previously glued on the bottle surface, compared with  $X$  read at the horizontal rule. From now on, we will study the complete flow dynamics instead of only the instantaneous Torricelli’s relation between speed and height.

Figure 7 exhibits the reduced speed  $v$  as a function of the time  $T$  elapsed since the initial height  $H_0 = 20.0\text{cm}$ . The continuous line is the best linear fit, excluding the last six points where instabilities at the pin-hole start to be noted. According to this fit, the complete flow spends a time  $T_0 = 526\text{s}$ . The real time, however, is smaller than that, something around  $450\text{s}$ , according to the last point in figure 7. For  $H < 5\text{cm}$  the flow at the pin-hole starts to present small intermittent bursts superimposed to the continuous, stable flow observed before. This phenomenon is more and more clearly observed as time goes by, until only isolated drops could be observed at the end, below the residual height  $R$ , with no longer traces of any underlying continuous flow. This continuous-plus-intermittent way is more effective in draining off the bottle faster than it would be by following the previously observed pure-continuous flow, according to the straight line in figure 7. Also, Bernoulli’s theorem, even with our constant energy amount subtracted at the pin-hole, is no longer valid: it supposes one has a *stationary* flow. The theoretical treatment of this intermittency problem is a very difficult task, and many interesting complex phenomena could be observed experimentally [8]. No satisfactory analytical approach is available, although some computer simulations based on a stochastic simple model seems to capture the essential physical ingredients governing the phenomenon [9] — see [10] for a review.

Before the intermittent regime arises below  $H \approx 5\text{cm}$ , however, Bernoulli’s theorem

can be applied to our bottle flow, in order to compare the results with the dynamical experimental counterparts shown in figure 7. One also needs to include the condition that the liquid is not compressible, i.e.

$$aV = -A \frac{dH}{dT} , \quad (7)$$

where  $a$  and  $A$  are the cross sections already mentioned. The result is a simple differential equation, namely

$$\frac{dV}{dT} = -\frac{a}{A}g . \quad (8)$$

According to this result, the speed  $V$  (or  $v$ ) decreases linearly as time goes by, just as can be seen in the experimental plot, figure 7, before the intermittent regime starts to appear. Also according to this result, the complete draining time would be [5]

$$T_0 = \frac{AV_0}{ag} = \frac{A}{a} \sqrt{\frac{2(H_0 - R)}{g}} . \quad (9)$$

Taking our experimental values  $T_0 = 526\text{s}$ ,  $H_0 = 20.0\text{cm}$  and  $R = 2.05\text{cm}$ , adopting  $g \cong 9.8\text{m/s}^2$ , and considering a bottle diameter of  $9.5\text{cm}$ , we can estimate  $a$ . As already mentioned,  $a$  is not the cross section of the pin-hole, but that of the liquid vein (*vena contracta*). Indeed, equation (9) gives a diameter of  $1.8\text{mm}$  for the *vena contracta*, a little bit smaller than that of the pin-hole, namely  $2.0\text{mm}$ .

From now on, we will also use a lowercase, dimensionless, reduced variable

$$t = \frac{T}{T_0} , \quad (10)$$

when referring to the time. With this notation, the theoretical time evolutions of the reduced speed and height are simply

$$v = 1 - t , \quad (11)$$

and

$$h = (1 - t)^2 , \quad (12)$$

again independent of the particular (cylindrical) bottle actually used during the experiment. Both the experimental and theoretical values for these evolutions are shown in figure 8. Once more, deviations between theory and experiment are visible at the end, where the

flow starts to present an intermittent, non-stationary component. Note, in particular, that the final height ( $R \cong 2\text{cm}$  within the particular geometry of our bottle, corresponding to  $h = 0$  in figure 8) is actually reached faster than it would be according to the completely stable flow observed before (theoretical, continuous curve). The intermittency appearing at the end of the whole process helps to drain off the bottle faster.

### III — Two-Bottles Experiment

Considering now the system shown in figure 2, in applying condition (7) to the lowest bottle, one needs to take into account the extra amount of water coming from the upper bottle. It reads now

$$a(V' - V) = -A \frac{dH'}{dT} , \quad (13)$$

where the height  $H'$  and the speed  $V'$  refer to the lowest bottle, being related one to the other by the same modified Torricelli's law

$$V'^2 = 2g(H' - R) , \quad (14)$$

already verified in our first experiment, with  $R = 2.05\text{cm}$ . Both the bottles and the pin-hole tubes are identical. We define the reduced speed

$$u = \frac{V'}{V_0} , \quad (15)$$

analogously to equation (5). The time evolution of  $u$  follows the dimensionless differential equation

$$\left(1 + \frac{du}{dt}\right) u = 1 - t , \quad (16)$$

which is the same equation (13), after being simplified by using (14), (15), (9) and (11). Here,  $t$  is the same reduced time already defined in equation (10).

One can solve this equation numerically, by dividing the time interval  $0 \leq t \leq 1$  in  $N$  equal subintervals. The lowercase indices  $n = 0, 1, 2 \dots N$  represent the discretized times at the borders of these subintervals, i.e.  $t_n = n/N$ , while  $u_n$  represents the value of  $u$  at time  $t_n$ . The derivative in equation (16) will be replaced by the approximation

$$\frac{du}{dt} \approx \frac{u_{n+1} - u_n}{1/N} , \quad (17)$$



where  $1/N$  is the size of each subinterval. This approximation is as good as we want, provided a large value for  $N$  is adopted. In other words, the numerical solution we will obtain is exact in the sense that one always can keep the errors below any predefined tolerance, no matter how exacting is the user who chooses the degree of tolerance. Equation (17) stands for the derivative of  $u$  taken at the center of the subinterval, i.e. at time

$$t = \frac{n + 1/2}{N} , \quad (18)$$

which is the value replacing  $t$  at the right-hand side of equation (16). Accordingly, another approximation, namely

$$u \approx \frac{u_{n+1} + u_n}{2} , \quad (19)$$

will replace  $u$  outside the parenthesis at the left-hand side of (16). Then, by performing these replacements, equation (16) is transformed into a second degree equation for  $u_{n+1}$  whose solution is

$$u_{n+1} \approx \frac{1}{2N} [\sqrt{(2Nu_n - 1)^2 + 8(N - n) - 4} - 1] . \quad (20)$$

Now, one needs only to choose a convenient value for  $N$ , program this equation on a computer (or even a pocket calculator) and process it starting from  $u_0 = 1$  with  $n = 0$  at the right-hand side. The result will be the next value, i.e.  $u_1$ . Then, processing again the same program with this recently obtained value for  $u_1$  and  $n = 1$  at the right-hand side, one gets  $u_2$ . After repeating this process  $N$  times, one has the complete function  $u(t)$  along the interval  $0 \leq t \leq 1$ .

The lowest bottle is fed by the upper one, up to  $t = 1$  when the upper flow ceases. However, at this time the lowest bottle is still flowing with a reduced speed  $u^*$ . From this moment on, there is only one flowing bottle, as in our first experiment. Thus,  $u$  will decrease linearly, starting with  $u = u^*$  at  $t = 1$ , at the same constant rate observed in our first experiment, namely  $du/dt = -1$ . The bottom flow finally also ends at  $t = 1 + u^*$ . The value  $u^*$  is then a key parameter for our system. By running our program for equation (20) with  $N = 100$ , we obtained  $u^* = 0.54630191$ . Running it again with  $N = 1,000$ , we got  $u^* = 0.54629310$ , which means that  $N = 100$  is enough for an accuracy up to the fourth decimal figure, corresponding to a relative deviation of the order of  $10^{-5}$ , far below our experimental accuracy. By running the same program with  $N = 10,000$ , and once again with  $N = 100,000$ , which spends much less than one second on a PC, we obtained

$$u^* = 0.54629302 , \quad (21)$$

in both cases, meaning that this value is the exact one, at least up to the 8<sup>th</sup> decimal figure.

From equation (16) one can verify two limiting cases. First, near  $t = 0$ , one must have  $u \cong 1 - t^2/2$ . Note that the first derivative of  $u$  vanishes at  $t = 0$ , as it must be at the beginning because the outflux of the lowest bottle is exactly compensated by the influx coming from the top bottle (remember that both bottles start from the same initial heights  $H_0 = H'_0$ ). Second, in reaching  $t = 1$  from below, one has  $du/dt = -1$ , compatible with the one-bottle constant flow regime which will be installed from this moment on. Both limiting cases can also be observed in equation (20), namely  $u_1 \cong 1 - 1/2N^2$  and  $u_N - u_{N-1} \cong -1/N$ .

Alternatively, one can solve equation (16) analytically, by replacing  $u$  by  $\omega(1 - t)$ , which allows one to separate the variables  $t$  and  $\omega$  into different sides of the resulting equation, i.e.

$$\frac{dt}{1-t} = \frac{\omega d\omega}{1-\omega+\omega^2} \quad . \quad (22)$$

Performing the integration, one finds finally the ugly analytical solution

$$1-t = \frac{\exp\left(-\frac{\pi}{6\sqrt{3}}\right) \exp\left[-\frac{1}{\sqrt{3}} \operatorname{tg}^{-1}\left(\frac{2\omega-1}{\sqrt{3}}\right)\right]}{\sqrt{\omega^2 - \omega + 1}} \quad . \quad (23)$$

Going back to  $u = \omega(1 - t)$ , we have plotted  $u$  *versus*  $t$  from this solution, a task which requires much more computer time than the whole processing of the numerical solution (20). The resulting plot is indistinguishable from the one obtained numerically with  $N = 100$ . Also, by taking the limit  $t \rightarrow 1$  in this solution, we obtained

$$u^* = \exp\left(-\frac{\pi}{3\sqrt{3}}\right) \quad , \quad (24)$$

in complete agreement with the numerically obtained value (21).

The above paragraphs of this section concern what says the theory about our two-bottles experiment. We have also measured the successive heights  $H'$ , from which we obtained the reduced speed

$$u = \sqrt{\frac{H' - R}{H_0 - R}} \quad , \quad (25)$$

as a function of the time. The result is plotted in figure 9, together with the theoretical continuous curve. The point denoted by P corresponds to  $u^*$ , at  $t = 1$ . Our experimental value is

$$u^* = 0.535 \quad , \quad (26)$$

obtained by interpolating the two experimental points which are nearest to  $t = 1$ . The relative deviation between theory and experiment, concerning this value, is only 2%. In performing the experiment, we have been careful enough to interrupt the upper flow as soon as the top bottle reaches the residual level  $R \cong 2cm$ . After this, only isolated drops would feed the lowest bottle, but they were not included in our one-bottle, continuous flow analysis which ends at the residual level  $R$ . Thus, this further flow from isolated drops cannot be included now into our two-bottles experiment.

Note again deviations at the end of the flow, similar to those already appeared in the one-bottle experiment: the last points are located below the theoretical straight line, because the intermittent flow regime accelerates the draining off. Note also the smaller deviations appearing just before  $t = 1$ , now the experimental points being *above* the theoretical curve. This is so because the accelerated draining off of the top bottle, due to the intermittent flow occurring just before  $t = 1$ , feeds water into the lowest bottle *faster* than it would be according to the theoretical, continuous flow.

## IV — Conclusions

We performed a simple experiment on elementary hydrodynamics, verifying the validity of the so-called Torricelli's law, equation (1), relating the height  $H$  of the free surface of water in a bottle with the speed  $V$  at which the liquid is squirted through a pin-hole. The speeds are indirectly measured through the horizontal range  $X$  (figure 1). In order to take energy losses into account, a simple correction was introduced into this law: equation (3) replaces the original form (1). According to our experimental observations (figures 3 and 4) the residual height  $R$  subtracted from  $H$  in order to take energy losses into account is a constant during all the flow. We have also measured the dynamical evolution of this system, i.e. the dependences of both  $H$  and  $V$  on the time  $T$ , comparing the results with the theoretical framework of Bernoulli's theorem. We performed also another experiment, using two equal bottles, the liquid squirted by the first feeding the second one. In this case, we got a non-trivial differential equation from the same framework. It was solved both numerically and analytically, and the results were compared with the experimental dynamical evolution.

We believe such simple experiments could be reproduced by students, in order to better understand elementary hydrodynamic concepts. Also, other similar systems — e.g.

the ones quoted as examples, exercises and problems in references [5–7] — could be experimented along the same lines presented here. In such experiments, the fixed parameters characterizing each bottle are: 1) the initial height  $H_0$ ; 2) the initial speed  $V_0$ ; 3) the residual height  $R$ , experimentally obtained through the plot  $V^2$  versus  $H$  (figures 3 and 4); and 4) the total time  $T_0$  required to drain off the bottle, experimentally obtained through the plot  $V$  versus  $T$  (figure 7). On the other hand, the dynamical variables depending on the time  $T$  during the flow are:  $H$  and  $V$  for our one-bottle experiment (figure 1); or  $H'$  and  $V'$  for our two-bottles experiment (figure 2). Nevertheless, neither the expressions of these dynamical variables as functions of the time nor the mathematical relation between themselves would depend on the particular geometry of the cylindrical bottles and pin-holes actually used in the experiments. In order to stress this independence, we adopted the reduced, dimensionless, bottle independent variables  $h$ ,  $v$ ,  $t$  and  $u$  defined by equations (4), (5), (10) and (15). According to these dimensionless variables, the theoretical equations (6), (11), (12), (20) and (23) can be experimentally tested with other bottles, different from those we used here. We also believe the students must be introduced as often as possible to the very good scientific practice of using universal variables, independent of particular implementations or parameters.

This work was partially supported by brazilian agencies CAPES, CNPq and FAPERJ. One of us (PMCO) is grateful to the late professor Pierre Henrie Lucie from whom he learned how to use simple systems and experiments in order to teach Physics. Other (AD) is grateful to J.S. Sá Martins and Nivaldo A. Lemos, for enlightening discussions regarding the subject.

## References

- [1] A. Sommerfeld, *Vorlesungen über Theoretische Physik*, 2nd edition, DWB, Wiesbaden (1947) — Chapter III.
- [2] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, vol. I (exercises), Addison-Wesley, Reading, Massachusetts (1964) — exercise 4-12.
- [3] E.A. Hylleraas, *Mathematical and Theoretical Physics*, vol. I, Wiley-Interscience, New York (1970) — part II, Chapter 27.
- [4] P.A. Tipler, *Physics*, vol. I, Worth Publisher, New York (1976) — Chapter 13, exercise 26.
- [5] P.H. Lucie, *Física Básica*, vol. 2, Campus, Rio de Janeiro (1980) — Chapter 7, exercises 1R, 3R, 4R, 3, conceptual questions 2, 3, 5, problems 1–7, 11 and 14.
- [6] R.M. Eisberg, *Physics: Foundations and Applications*, McGraw-Hill, New York (1981)

- Chapter 16, example 9 and problems 42–45.
- [7] H.M. Nussenzveig, *Física Básica*, vol. 2, Edgar Blücher, São Paulo (1983) — Chapter 2, exercises 1–3, 6 and 7.
- [8] J.C. Sartorelli, W.M. Gonçalves and R.D. Pinto, *Phys. Rev.* **E49**(5), 3963-3975 (1994); X.D. Shi, M.P. Brenner and S.R. Nagel, *Science* **265**, 219-222 (1994); T.J.P. Penna, P.M.C. de Oliveira, J.C. Sartorelli, W.D. Gonçalves and R.D. Pinto, *Phys. Rev.* **E52**(3), R2168-R2171 (1995); R.D. Pinto, W.M. Gonçalves, J.C. Sartorelli and M.J. de Oliveira, *Phys. Rev.* **E52**(6), 6896-6899 (1995); M.S.F. da Rocha, J.C. Sartorelli, W.M. Gonçalves and R.D. Pinto, *Phys. Rev.* **E54**(3), 2378-2383 (1996); J.G. Marques da Silva, J.C. Sartorelli, W.M. Gonçalves and R.D. Pinto, *Phys. Lett.* **226A**, 269-274 (1997); W.M. Gonçalves, R.D. Pinto, J.C. Sartorelli and M.J. de Oliveira, *Physica* **A257**, 385-389 (1998); R.D. Pinto, W.M. Gonçalves, J.C. Sartorelli, I.L. Caldas and M.S. Baptista, *Phys. Rev.* **E58**(3), 4009-4011 (1998); S.R. Nagel, *Am. J. Phys.* **67**(1), 17-25 (1999); and references therein.
- [9] P.M.C. de Oliveira and T.J.P. Penna, *J. Stat. Phys.* **73**(3/4), 789-798 (1993); P.M.C. de Oliveira and T.J.P. Penna, *Int. J. Mod. Phys.* **C5**(6), 997-1006 (1994); A.R. Lima, T.J.P. Penna and P.M.C. de Oliveira, *Int. J. Mod. Phys.* **C8**(8), 1073-1080 (1997).
- [10] S. Moss de Oliveira, P.M.C. de Oliveira and D. Stauffer, *Evolution, Money, War and Computers: Non-Traditional Applications of Computational Statistical Physics*, Teubner, Stuttgart-Leipzig (1999) — Chapter 3.

### Figure Captions

- Fig. 1 Experimental apparatus used in our test of Torricelli's Law. Successive values of the height  $H$  were read from marks previously glued on the bottle surface, 0.5cm apart from each other. The successive speeds of the water squirted at the pin-hole were indirectly measured through the ranges  $X$  read at the horizontal rule. Successive times  $T$  were also measured during the flow.
- Fig. 2 Two-bottles experiment, where the water squirted from the first is caught by the second through a funnel. Successive heights  $H'$  were measured as a function of the time.
- Fig. 3 Squared range  $X^2$  versus height  $H$  for the one-bottle experiment shown in figure 1. Our estimated error bars are represented by the symbol's size. The continuous line is the best linear fit for the experimental data. The inner diameter of the bottle is 9.5cm, and that of the pin-hole tube is 2.0mm. The initial height is  $H_0 = 20.0\text{cm}$ , according to which we measured the initial horizontal range  $X_0 = (28.0 \pm 0.3)\text{cm}$ .

- Fig. 4 The same as in figure 3, now for a different bottle with a different pin-hole. The inner diameter of the bottle is 5.1cm, while that of the pin-hole tube is 3.5mm. The initial height is  $H_0 = 42.0\text{cm}$ , and the corresponding initial horizontal range is  $X_0 = (44.0 \pm 0.3)\text{cm}$ .
- Fig. 5 Experimental speed *versus* height. Different symbols correspond to the same experiment performed with different bottles with different pin-holes, according to figures 3 (circles) and 4 (crosses). By adopting the dimensionless, reduced variables  $v$  and  $h$ , defined by equations (4) and (5), our results become independent of the particular geometrical dimensions of the bottle and the pin-hole: both data sets could be superimposed into the same, universal, bottle-independent curve. The continuous line corresponds to the theoretical Torricelli's law, equation (6).
- Fig. 6 Same experimental data of figure 5. Now, the continuous line is a linear fit for the circles, excluding the last points of the flow where instabilities and oscillations appear. The slopes measured for each different bottle must be compared with the exponent  $1/2$  of the square root in theoretical equation (6).
- Fig. 7 Time evolution of the speed  $v$ . The continuous line is the best fit excluding heights  $H < 5\text{cm}$ , below which an intermittent flow regime starts to be observed.
- Fig. 8 Time evolution of the dimensionless speed and height, as functions of the dimensionless time, equation (10). Experimental and theoretical values agree very well with each other, except for the last points where the flow is no longer stationary.
- Fig. 9 Experimental time evolution of the dimensionless speed  $u$  of the lowest bottle, as a function of the dimensionless time, in the two-bottles experiment (figure 2). The continuous line comes from the theory.

This figure "fig1.jpg" is available in "jpg" format from:

<http://arXiv.org/ps/cond-mat/0005428v1>

This figure "fig2.jpg" is available in "jpg" format from:

<http://arXiv.org/ps/cond-mat/0005428v1>















